OPTIMAL EXTRACTION OF A RESOURCE WITH EXTRACTION COSTS

$$\max \ W = \int_0^{\infty} U(C_t) e^{-\rho t} dt$$

$$\text{s.t.} \quad \dot{K}_t = Q(K_t, R_t) - C_t - G(R_t, S_t)$$
$$\dot{S}_t = F(S_t) - R_t$$

$$(1) \ \mathcal{H}_c = U(C_t) + \mu_t^K \left[Q(K_t, R_t) - C_t - G(R_t, S_t) \right]$$
$$\mu_t^S \left[F(S_t) - R_t \right]$$

$$\frac{\partial \mathcal{H}_c}{\partial C_t} = U'(C_t) - \mu_t^K = 0 \Rightarrow \mu_t^K = U'(C_t) \quad [A]$$

$$\frac{\partial \mathcal{H}_c}{\partial R_t} = \mu_t^K (Q_R - G_R) - \mu_t^S = 0 \Rightarrow \frac{\mu_t^S}{\mu_t^K} = Q_R - G_R \quad [B]$$

$$\Rightarrow \mu_t^S = U'(C_t)(Q_R - G_R)$$

$$\dot{\mu}_t^K = \rho \mu_t^K - \frac{\partial \mathcal{H}_c}{\partial K} = \rho \mu_t^K - \mu_t^K Q_K \Rightarrow \frac{\dot{\mu}_t^K}{\mu_t^K} = \rho - Q_K \quad [C]$$

$$\dot{\mu}_t^S = \rho \mu_t^S - \frac{\partial \mathcal{H}_c}{\partial S} = \rho \mu_t^S + \mu_t^K G_S - \mu_t^S F'(S_t) \quad [D]$$

$$\Rightarrow \frac{\dot{\mu}_t^S}{\mu_t^S} = \rho + \frac{\mu_t^K}{\mu_t^S} G_S - F'(S_t) = \rho + \frac{G_S}{Q_R - G_R} - F'(S_t)$$

INTERPRETATIONS OF NECESSARY CONDITIONS:

Condition [A] says that the last unit of capital consumed must increase instantaneous utility by the same amount that it decreases its present value contribution to future welfare, defined as the shadow value of capital.

Condition [B] says that we must be indifferent between leaving an extra unit of the resource in the ground, or extracting it to increase the stock of capital.

Condition [C] says that the rate of increase of the shadow value of capital (capital gain) and its present contribution to output (dividend) must be equal to the discount rate.

Condition [D] says the same as [C], except that with the resource, an extra unit of its stock, by reducing extraction costs, allows for the accumulation of more capital, on top of its direct effect $F'(S_t)$. Note that $G_s < 0$!

(2) STEADY STATE:

$$\dot{\mu}_t^K = \dot{\mu}_t^S = \dot{K}_t = \dot{S}_t = 0.$$

$$\Rightarrow \quad C = Q(K,R) - G(R,S)$$
$$R = F(S)$$
$$Q_K = \rho$$

$$F'(S) - \frac{G_S}{Q_R - G_R} = \rho$$

(3) NON-RENEWABLE RESOURCE:

$$\Rightarrow F(S) = F'(S) = 0.$$

If $R>0$ is necessary for $Q>0$, then there is no steady state.