Non-renewable extraction when costs depend on remaining stock level

$$\max_{q(t)\geq 0} \int_0^T e^{-\delta t}\left[p(t)-c(S(t))\right]q(t)\,dt$$

$$\text{s.t.} \quad \dot{S}(t) = -q(t), \quad 0 \leq p(t) < \infty$$

$$S(0) = S_0$$

$\lambda(t) \equiv \mu(t)e^{-\delta t}$

$\Rightarrow \dot{\lambda} = \dot{\mu}e^{-\delta t} - \delta\mu e^{-\delta t}$

a) $$\mathcal{H}_c = \left[p(t) - c(S(t))\right]q(t) - \mu(t)\,q(t)$$

$$\frac{\partial \mathcal{H}_c}{\partial q(t)} = p(t) - c(S(t)) - \mu(t) = 0.$$

$\Rightarrow \mu(t) = p(t) - c(S(t)) \equiv$ The marginal contribution of the resource stock to present profits.

$$\dot{\mu}(t) - \delta\mu(t) = -\frac{\partial \mathcal{H}_c}{\partial S(t)} = c'(S(t))\,q(t)$$

$$\Rightarrow \frac{\dot{\mu}(t) - c'(S(t))\,q(t)}{\mu(t)} = \delta$$: The flow of benefit from a marginal unit of the resource is composed of its capital gain ($\dot{\mu}(t)$) and its total contribution to lower extraction costs ($c'(S)q$).

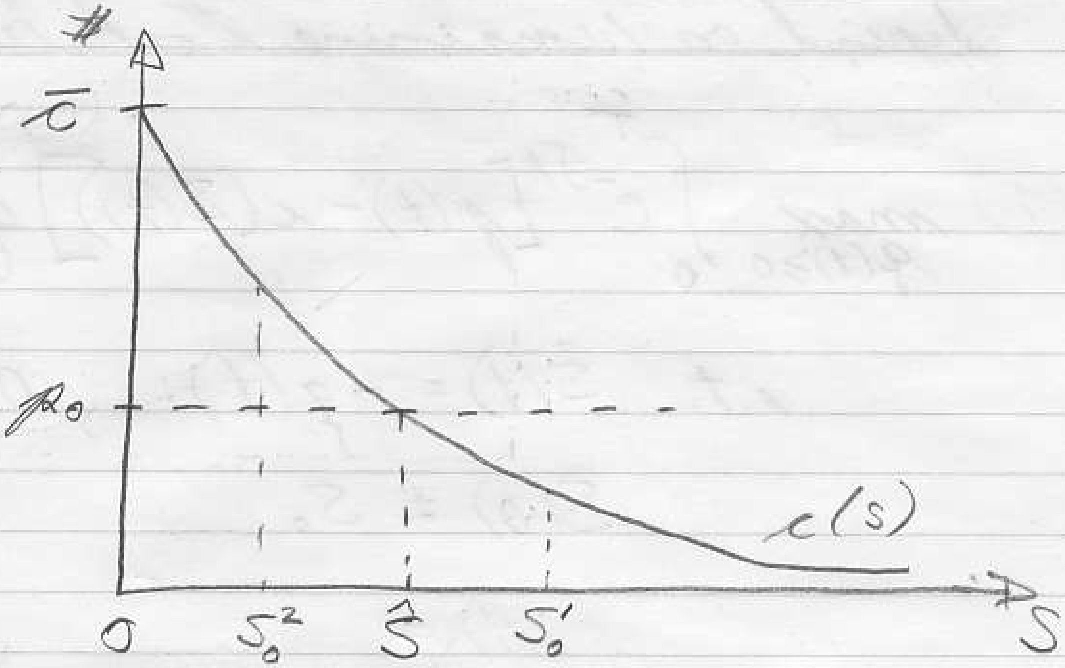
We have: $\dot{\mu} = \dot{p} + c'(S)q \Rightarrow$ The change in value of the resource must follow the change in its marginal contribution to profits.

Note how the total contribution to lower extraction costs is cancelled out by the change in value of the resource from the cost side.

$$\Rightarrow \frac{\dot{p}}{p} = \delta \Rightarrow \frac{\dot{p}}{p - c(S)} = \delta$$

b) $\dot{p} = 0 \Rightarrow$ There is no capital gain anymore from leaving the resource in the ground. It is thus better to extract all at once <u>until</u> $p = c(S)$.

Let p_0 be the constant price.



(i) If $S_0 = S_0' > \hat{S}$, then extract $S_0 - \hat{S}$ instantly and stop extraction.

$$\Rightarrow q(0) = +\infty$$

(ii) If $S_0 = S_0^2 < \hat{S}$, then no extraction ever takes place as the marginal cost is always above the price.

This example highlights the role of increasing resource price in order to slow down the extraction rate <u>initially</u>. But it also increases the extraction rate later.

c) $\dot{p}/p = \gamma$:

Along the interior optimal path, we must have:

$$\frac{\dot{p}}{p - c(S)} \equiv \frac{\dot{p}/p}{1 - c(S)/p} \equiv \frac{\gamma}{1 - c(S)/p} = \delta$$

(i) This suggests that if, at $t = 0$, we have

$$\frac{\gamma}{1 - c(S_0)/p_0} < \delta,$$ then extract

instantly amount $S_0 - S_0'$ s.t.

$$\frac{\gamma}{1 - c(S_0')/p_0} = \delta.$$

$\Rightarrow$ $q(0) = +\infty$. and follow the interior optimal path thereafter.

(ii) If $\frac{\gamma}{1 - c(S_0)/p_0} > \delta$, then leave

the resource in the ground until the increase in price reestablishes equality, i.e. start extracting at future date t' characterized by:

$q(t) = 0$ for $t \in (0, t')$ s.t. $\frac{\gamma}{1 - c(S_0)/p(t')} = \delta$.

After t', extract according to the rule $\frac{\gamma}{1 - c(S_0)/p(t)} = \delta$.

(iii) Note that if $\gamma \geq \delta$, then

$$\frac{\gamma}{1 - \frac{c(S_0)}{p(t)}} \geq \delta \quad \text{regardless of}$$

the values of S_0 and $p(t)$. The resource is left unexploited because the increase in its price provides a better return than the alternative best return δ.

If T is finite, then $q(t) = 0$ for $t < T$

and $q(T) = +\infty$ and $S_0 - S'$ is extracted at $t = T$ such that

$c(S') = p(T)$.