2) Pollution Discharge in a Lake

a) $$\max_{E(t)} J = \int_0^{\infty} -\left[\alpha(K - E(t))^2 + \gamma S(t)^2\right] e^{-\rho t}\, dt$$

$$\text{s.t. } \dot{S} = -\beta S(t) + E(t)$$

$$H = -\left[\alpha(K - E(t))^2 + \gamma S^2(t)\right] e^{-\rho t} + \lambda(t)\left(E(t) - \beta S(t)\right)$$

$$\frac{\partial H}{\partial E(t)} = +2\alpha(K - E(t)) e^{-\rho t} + \lambda(t) = 0 \qquad [*]$$

$$\Rightarrow \lambda(t) = \underline{-2\alpha(K - E(t))}\, e^{-\rho t} < 0$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial S(t)} = -\left[-2\gamma S(t) e^{-\rho t} - \lambda(t)\beta\right]$$

let $\lambda(t) = \mu(t) e^{-\rho t} \Rightarrow \dot{\lambda} = \dot{\mu} e^{-\rho t} - \rho\mu e^{-\rho t}$

$$= (\dot{\mu} - \rho\mu)\, e^{-\rho t}$$

$$\Rightarrow \dot{\mu} - \rho\mu = 2\gamma S + \mu\beta$$

$$\Rightarrow \frac{\dot{\mu} - 2\gamma S - \mu\beta}{\mu} = \rho \qquad [**]$$

[*] Says that if discharge is increased by one ~~marginal~~ unit, then there is an instantaneous PV gain of $2\alpha(K - E(t)) e^{-\rho t}$ due to lower cost of pollution reduction.

But there is a loss of $\lambda(t)$ because the pollution stock increases faster by one unit.

[**] Denote the three types of returns from the pollution stock:

① The change in shadow value $\dot{\mu}$

② The marginal pollution damage $-2\gamma S$

③ The gain from a reduction in the stock $-\mu\beta$.

e) In S-S, we have:

$$\dot{\mu}=0, \quad \dot{S}=0.$$

$$\Rightarrow \quad S^{ss}=\frac{E^{ss}}{\beta}$$

$$\mu^{ss}=-2\alpha(K-E^{ss})$$

$$(\rho+\beta)\mu^{ss}=-2\gamma S$$

$$\Rightarrow (\rho+\beta)\,2\alpha(K-E^{ss})=2\gamma S^{ss}$$

$$\Rightarrow (\rho+\beta)\alpha(K-\beta S)=\gamma S$$

$$(\rho+\beta)\frac{\alpha}{\gamma}\left(\frac{K}{S}-\beta\right)=1$$

$$\frac{K}{S}=\frac{\gamma}{\alpha(\rho+\beta)}+\beta \quad \ldots$$