The natural growth of a fish population

Suppose that in the absence of fishing activities, the fish stock at time t is given by

$$S(t) = \frac{\bar{S}}{1 + \alpha e^{-\rho t}} \text{ where } \alpha \equiv \frac{\bar{S} - S_0}{S_0}. \tag{1}$$

a) Show that (1) implies that the natural rate of change as a function of the stock size $G(S)$ is given by

$$G(S) = \rho \left(1 - \frac{S}{\bar{S}}\right) S. \tag{2}$$

(Hint: Derive $\dot{S}$ from (1) then substitute for $\alpha \exp^{-\rho t}$ by rearranging (1).)

b) Find the implied stock sizes at $t = 0$ and $t \to \infty$. How does the proportional growth rate of the resource $\dot{S}/S$ vary with stock size? Where is it maximized? At what stock size is the absolute growth rate $\dot{S}$ maximized? Sketch the evolution of the fish stock as a function of time on a graph. Sketch the natural rate of change as a function of the stock size. Discuss.

c) Let us now introduce fishing activities in the form of a constant harvest rate y. The rate of change of the resource stock is now given by $\dot{S} = G(S) - y$. Determine the value $\bar{y}$ which corresponds to the highest sustainable harvest rate in terms of parameter value ρ and $\bar{S}$. What happens when $y = \rho\bar{S}/2$? When $y = \rho\bar{S}/8$? (Clue: Use a graphic to think about this problem. A sustainable harvest rate is one where $\dot{S} = 0$.)