J. XR 4.5)

According to Theorem 3.4, the cost fctn of a CRS production technology can be expressed as:

$$c(\vec{w}, y) = y\, c(\vec{w}, 1).$$

If $p(q)$ is the inverse demand, the LR equil. is such that

$$\Pi_i = p\left(\sum_{j=1}^{J} y_j\right) y_i - y_i\, c(\vec{w}, 1) = 0, \quad \forall i$$

$$\Rightarrow \left[p\left(\sum_{j=1}^{J} y_j\right) - c(\vec{w}, 1)\right] y_i = 0$$

Hence, the long-run equil. is given by

$$p\left(\sum_{j=1}^{J} y_j\right) = c(\vec{w}, 1).$$

$$\Rightarrow \sum_{j=1}^{J} y_j = q(c(\vec{w}, 1)).$$

The LR equil. is thus characterized by a total output equal to $q(c(\vec{w}, 1))$, regardless of the number of firms producing or the distribution of output across firms.

J.&R. 4.6)

Assumptions:

- Competitive industry
- total cost: $c_j(q) = aq + b_j q^2$

a) $b_j > 0$.

The competitive equilibrium is characterized by $MC = p$.

$\Rightarrow c_j'(q) = a - 2 b_j q_j = p(\sum_i q_i)$

$\Rightarrow q_j = \dfrac{a - p(\sum_i q_i)}{2 b_j}$

Since $a - p(\sum q_i)$ is the same for all firms, those with larger b_j will produce less in the competitive equilibrium.

b) $b_j < 0$:

We have:

$\Pi_j = p q_j - (a q_j + b_j q_j^2)$

$\Rightarrow \dfrac{\partial \Pi_j}{\partial q_j} = p - a - 2 b_j q_j$

$\dfrac{\partial^2 \Pi_j}{\partial q_j^2} = -2 b_j > 0.$

The second-order condition for a maximum is not respected. This

means that if marginal profits can be positive, firms will want to keep on producing more.
Note that production costs are nil at

$$a + b_i q_i = 0 \Rightarrow q_i = \frac{-a}{b_i}.$$

Such a production function does not make much sense when $b_i < 0$.

J&R 4.8)

Assumptions:

- Cournot competition $J=2$
- $p = a - b(q_1+q_2)$
- $c_1(q_1) = c_1 q_1$
- $c_2(q_2) = c_2 q_2 \qquad c_1 < c_2$

In order to solve for the Nash equilibrium, we must first find the optimal output level of firm i for any output choice of firm j.

Problem of firm 1:

$$\max_{q_1} \pi_1 = p q_1 - c_1 q_1$$

$$= [a - b(q_1+q_2)] q_1 - c_1 q_1$$

FOC: $$\frac{\partial \pi_1}{\partial q_1} = a - b q_2 - 2b q_1^* - c_1 = 0$$

$$\Rightarrow \boxed{q_1^*(q_2) = \frac{a - b q_2 - c_1}{2b}}$$

This is firm 1's profit max. output level for any given q_2.

Similarly, we have:

$$\boxed{q_2^*(q_1) = \frac{a - b q_1 - c_2}{2b}}$$

In the Nash equil., both firms must be maximizing profits given the other firm's choice; that is,

$$q_1^N = q_1^*(q_2^*(q_1^*)) \text{ and } q_2^N = q_2^*(q_1^*(q_2^*))$$

Substituting $q_2^*(q_1)$ into $q_1^*(q_2)$, we get:

$$q_1^N = \frac{a-c_1}{2b} - \frac{1}{2}\left[\frac{a-c_2}{2b} - \frac{q_1^N}{2}\right]$$

$$= \frac{a-c_1}{2b} - \frac{a-c_2}{4b} + \frac{q_1^N}{4}$$

$$\Rightarrow q_1^N = \frac{4}{3}\left[\frac{a-c_1}{2b} - \frac{a-c_2}{4b}\right] = \frac{1}{3b}\left(2(a-c_1)-(a-c_2)\right)$$

$$\boxed{q_1^N = \frac{1}{3b}(a-2c_1+c_2)}$$

Analogously for firm 2:

$$\boxed{q_2^N = \frac{1}{3b}(a-2c_2+c_1)}$$

q_1^N an q_2^N are the NE output levels.

Clearly, if $c_1 < c_2$, we have

$$q_2^N < q_1^N.$$

$q^N \equiv q_1^N + q_2^N = \frac{1}{3b}(2a-c_1-c_2)$ = total NE output.

$$\Rightarrow p^N = a - \frac{1}{3}(2a-c_1-c_2) =$$

$p^N = \frac{1}{3}(a+c_1+c_2)$: This is the NE price.

$$\Rightarrow \pi_1^N = \left[\frac{1}{3}(a+c_1+c_2) - c_1\right]\left[\frac{1}{3b}(a-2c_1+c_2)\right]$$

$\boxed{\pi_1^N = \frac{(a-2c_1+c_2)^2}{9b}}$ = firm 1 NE profit.

Analogously for firm 2:

$$\boxed{\pi_2^N = \frac{(a-2c_2+c_1)^2}{9b}}$$

$$\Rightarrow \pi_1^N > \pi_2^N \text{ if } c_1 < c_2.$$

J. & R. 4.9)

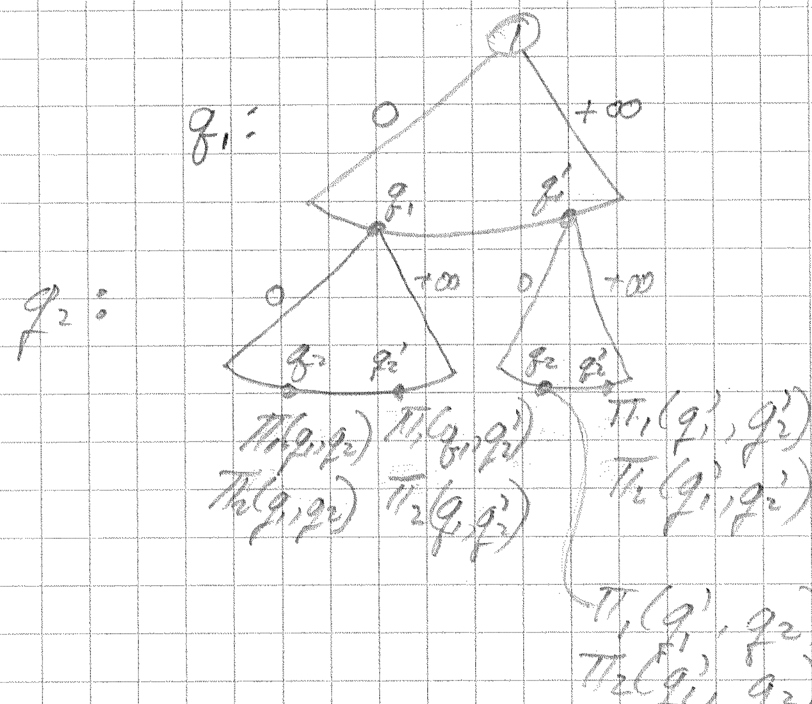
Stackelberg Duopoly ⇒ sequential game

$p = 100 - (q_1 + q_2)$

$c_1 = 10\, q_1$

$c_2 = q_2^2$

a) Firm 1 is leader.
Firm 2 is follower.



Using <u>backward induction</u>, we first solve for firm 2's reaction fctn: $q_2^*(q_1)$.

$$\max_{q_2} \Pi_2 = p(q_1 + q_2)\, q_2 - c_2(q_2)$$

$$= [100 - (q_1 + q_2)]\, q_2 - q_2^2$$

$$\frac{\partial \Pi_2}{\partial q_2} = 100 - q_1 - 2q_2^* - 2q_2^* = 0$$

$$\Rightarrow \boxed{q_2^*(q_1) = \frac{100 - q_1}{4}}$$ Reaction function of firm 2.

As a leader, firm 1 anticipates that for any output level q_1, firm 2's output level will be $q_2^*(q_1)$. Hence, the problem of firm 1 can be expressed as:

$$\max_{q_1, q_2} \Pi_1 = p(q_1+q_2) - c(q_1)$$

$$\text{s.t. } q_2 = \frac{100-q_1}{4}$$

Substituting, we get:

$$\max_{q_1} \Pi_1 = \left[100 - q_1 - \left(\frac{100-q_1}{4}\right)\right] q_1 - 10q_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = 100 - q_1^L - 25 + \frac{q_1^L}{4} - q_1^L + \frac{q_1^L}{4} - 10 = 0$$

$\Rightarrow \boxed{q_1^L = 50}$ Firm 1's leader equil. output level.

$\Rightarrow \boxed{q_2^F = 12.5}$ Firm 2's follower equil. output level.

$\Rightarrow p^S = 100 - 62.5 = \boxed{37.5}$ Stackelberg equil. price.

$\Rightarrow \Pi_1^L = (37.5 - 10)50 = 1375$ Leader's profit

$\Pi_2^F = 37.5 \cdot 12.5 - (12.5)^2 = 312.5$ Follower's profit

b) Firm 2 is leader.
Firm 1 is follower.

Firm 1's reaction fctn:

$$\max_{q_1} \Pi_1 = (100 - q_1 - q_2) q_1 - 10 q_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = 100 - 2q_1^* - q_2 - 10 = 0$$

$$\Rightarrow \boxed{q_1^*(q_2) = 45 - \frac{q_2}{2}}$$

Problem of firm 2 as leader:

$$\max_{q_2} \Pi_2 = \left(100 - \left(45 - \frac{q_2}{2}\right) - q_2\right) q_2 - q_2^2$$

$$\frac{\partial \Pi_2}{\partial q_2} = 100 - 45 - 2\left(\frac{q_2}{2} - q_2\right) - 2q_2^L = 0$$

$$\Rightarrow \boxed{q_2^L = \frac{55}{3} = 18.3}$$ Firm 2's leader equil. output level

$$q_1^F = 45 - \frac{55}{6} = \boxed{\frac{215}{6} = 35.8}$$ Firm 1's follower equil. output

$$\Rightarrow p^{S'} = 100 - \frac{55}{3} - \frac{215}{6} = 45.83$$

$$\Rightarrow \Pi_2^L = \left(45.83 - \frac{55}{3}\right) \frac{55}{3} = 504.1$$

$$\Pi_1^F = (45.83 - 10) \frac{215}{6} = 1284$$

c) We have:

$\pi_1^L > \pi_1^F$

$\pi_2^L > \pi_2^F$

Both firms prefer to be leaders.

d) Cournot-Nash Equil.

In the Nash equil. both firms must be on their reaction fctns:

$$q_1^N = 45 - \frac{q_2^N}{2}$$

$$q_2^N = 25 - \frac{q_1^N}{4}$$

$$\Rightarrow q_1^N = 45 - \frac{25}{2} + \frac{q_1^N}{8} \Rightarrow q_1^N = \frac{8}{7}\left(\frac{65}{2}\right) = \frac{260}{7} = 37.1$$

$$\Rightarrow q_2^N = 25 - \frac{65}{7} = \frac{110}{7} = 15.7$$

$$\Rightarrow p^N = 100 - \left(\frac{370}{7}\right) = \boxed{\frac{330}{7} = 47.1} \text{ NE price.}$$

$$\Rightarrow \Pi_1^N = \left(\frac{330}{7} - 10\right)\frac{260}{7} = 1380$$

$$\Pi_2^N = \left(\frac{330}{7} - \frac{110}{7}\right)\frac{110}{7} = 494$$

We have:

$$\Pi_1^N > \Pi_1^L > \Pi_1^F$$

$$\Pi_2^L > \Pi_2^N > \Pi_2^F$$

Firm 1 prefers the simultaneous choice than to be a leader.

Firm 2 always prefers to be a leader.

G & R. 4.10)

$$C(q_j) = c q_j \quad , \quad j = 1,2$$

$$p = a - b(q_1 + q_2)$$

a) As derived in section 4.2.1, the reaction fctn of firm 2 is:

$$q_2^*(q_1) = \frac{a-c}{2b} - \frac{q_1}{2}$$

The problem of firm 1 as leader is thus:

$$\max_{q_1} \Pi_1 = p(q_1 + q_2^*(q_1))q_1 - cq_1$$

$$= a - b\left[q_1 + \frac{a-c}{2b} - \frac{q_1}{2}\right] q_1 - cq_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = a - 2bq_1^L - \frac{a-c}{2} + bq_1^L - c = 0$$

$$\Rightarrow \boxed{q_1^L = \frac{a-c}{2b}}$$ Firm 1's output as leader.

$$\Rightarrow q_2^F = \frac{a-c}{2b} - \frac{1}{2}\left(\frac{a-c}{2b}\right) = \boxed{\frac{a-c}{4b}}$$ Firm 2's output as follower.

$$\Rightarrow Q^S = q_1^L + q_2^F = \frac{3}{4}\left(\frac{a-c}{b}\right)$$ Total industry output with Stackelberg duopoly.

$$\Rightarrow p^S = a - bQ^S = a - \frac{3}{4}(a-c)$$

$$\Rightarrow p^S = \frac{1}{4}(a+3c)$$

$$\Rightarrow \Pi_1^L = \left[\frac{1}{4}(a+3c) - c\right]\left(\frac{a-c}{2b}\right) = \boxed{\frac{(a-c)^2}{8b}}$$ Leader's profits

$$\Pi_2^F = \left(\tfrac{1}{4}(a+3c)-c\right)\left(\frac{a-c}{4b}\right) = \boxed{\frac{(a-c)^2}{16b}}$$ Follower's profits

In the Cournot-Nash equilibrium, we found in section 4.2.1 with $J=2$:

$$\Pi_1^N = \Pi_2^N = \frac{(a-c)^2}{9b} \text{ and } q_1^N = q_2^N = \frac{a-c}{3b}$$

Hence, $\Pi^L > \Pi_1^N > \Pi^F$.

Both firms would thus prefer to be leaders.

b) If both firms act as a leader, then each chooses its output level assuming the other firm to react to it. Hence, each firm's output level is:

$$q_1 = q_2 = q^L = \frac{a-c}{2b}$$

$$\Rightarrow Q = \frac{a-c}{b} \Rightarrow p = a-(a-c) = c$$

$$\Rightarrow \Pi_1 = \Pi_2 = 0$$

This equilibrium is not a Nash equilibrium because each firm's belief about what the other is doing is not consistent with what it actually does. It is as if each firm were making a mistake about the sequence of decisions.

On the other hand, the equilibrium has the flavour of a Bertrand equilibrium. Could it be that attempts to be a leader unravels to a Bertrand-Nash competitive structure?

J.&R. 4.11)

Assumptions:
- Cournot competition with J firms
- $c(q) = k + cq \Rightarrow k$ is fixed cost
- $p = a - b\sum_i q_i$

a) Short-run Nash equil. (NE):

Following the same procedure as in exercise 14.8, we have:

$$q_1^* = \frac{1}{2b}\left(a - b\sum_{j=2}^{J} q_j - c\right)$$

Note that fixed cost k does not affect the profit max. output level for active firms.

Since firms are identical, we set $q_1^N = q_2^N = \dots = q_J^N \equiv q^N$.

$$\Rightarrow q^N = \frac{1}{2b}\left(a - b(J-1)q^N - c\right) \qquad 1 + \frac{J-1}{2} = \frac{J+1}{2}$$

$$\Rightarrow \boxed{q^N = \frac{a-c}{b(J+1)}}, \quad \forall j$$

$$\Rightarrow \boxed{Q^N = Jq^N = \left(\frac{J}{J+1}\right)\left(\frac{a-c}{b}\right)}$$ is total NE output

$$\Rightarrow p^N = a - bQ^N = a - \left(\frac{J}{J+1}\right)(a-c) \qquad 1 - \frac{J}{J+1} = \frac{1}{J+1}$$

$$\Rightarrow \boxed{p^N = \frac{a + Jc}{J+1}}$$ is NE price

$$\Rightarrow \pi^N = (p^N - c)q^N - k$$

$$= \left(\frac{a + Jc}{J+1} - c\right)\left(\frac{a-c}{b(J+1)}\right) - k \qquad \frac{J}{J+1} - \frac{J+1}{J+1} = \frac{-1}{J+1}$$

$$\boxed{\pi^N = \frac{1}{b}\left(\frac{a-c}{J+1}\right)^2 - k}$$ is NE profit per firm.

b) Long-run NE:

In the long-run, the number of firms J adjust so that profits are nil.

$$\pi^N(J) = \frac{1}{b}\left(\frac{a-c}{J+1}\right)^2 - k = 0$$

$$\Rightarrow \frac{a-c}{J+1} = \sqrt{bk}$$

$$\Rightarrow \boxed{\bar{J} = \frac{a-c}{\sqrt{bk}} - 1}$$

$\bar{J}$ represents the number of firms that insures that the NE in Cournot competition yields zero profits.

What happens when $k \to 0$?

J. & R. 4.12)

- Bertrand competition with $J=2$.
- $Q = \alpha - \beta p$

a) $c_1(q_1) = cq_1 + F$ for $q_1 \geq 0$.

$c_2(q_2) = cq_2$

i) If we assume that fixed cost F is incurred by firm 1 even when $q_1 = 0$, then the presence of F fixed cost has no impact on the NE as firms will try to undercut each other.

$\Rightarrow p_1^N = p_2^N = c$

$\Rightarrow \pi_1^N = -F$

$\pi_2^N = 0$

ii) If we assume that the fixed cost is incurred by firm 1 only if it is active, then we have:

$$c_1(q_1) = \begin{cases} cq_1 + F & \text{if } q_1 > 0 \\ 0 & \text{if } q_1 = 0. \end{cases}$$

$c_2(q_2) = cq_2$

A candidate for a NE would be that firm 2 chooses a price such that by undercutting, firm 1 makes negative profits; that is,

$$\pi_1 = [(p_2 - \varepsilon) - c][\alpha - \beta(p_2 - \varepsilon)] - F < 0.$$

This means that p_2^N must be such that:

$(p_2^N - c)(\alpha - \beta p_2^N) - F = 0$

and $p_1^N = p_2^N$ with $q_1^N = 0$.

assuming that $p_2^N \leq p^m$ (the monopoly price), firm 2 cannot increase profits with a lower price. Undercutting the price would drive profits to zero because then firm 1 captures the entire demand.

$\Rightarrow$ Firm 2 cannot do better by deviating.

As for firm 1,

$$p_1 < p_2^N \Rightarrow \Pi_1 < 0$$
$$p_1 > p_2^N \Rightarrow \Pi_1 = 0.$$

Firm 1 cannot do better by deviating.

(p_1^N, p_2^N) such that $(p_2^N - c)(\alpha - \beta p_2^N) = F$ is thus a NE. Note that we have

$$\Pi_1^N = 0$$
$$\Pi_2^N = F$$

Firm 2 takes advantage of firm 1's fixed costs to make positive profits.

NB I have assumed that firm 2 captures the entire demand when $p_1 = p_2$.

b)

i) $c_1(q_1) = cq_1 + F \quad , q_1 \geq 0$

$c_2(q_2) = cq_2 + F \quad , q_2 \geq 0.$

Again, the presence of fixed costs <u>even</u> when $q_i = 0$ does no alter the Bertrand NE. We get

$p_1^N = p_2^N = c$

and $\Pi_1^N = \Pi_2^N = -F$.

Competition is so harsh that firms cannot even cover their fixed cost.

ii) $c_i(q_i) = \begin{cases} cq_i + F & \text{if } q_i > 0 \\ 0 & \text{if } q_i = 0. \end{cases}$

If we assume that firms do not incur fixed costs when not producing, then a candidate for a NE is to choose (p_1^N, p_2^N) such that:

$(p_1^N - c)(\alpha - \beta p_1^N) = F \quad , q_1^N = \alpha - \beta p_1^N$

$p_2^N = p_1^N \quad , q_2^N = 0.$

Assuming that $p_1^N < p^M$ (monopoly price), firm 1 cannot increase profits with a lower price, while increasing its price drives output to zero.

Firm 2 would make negative profits with $p_2 < p_1^N$ and zero profit with $p_2 > p_1^N$.

$\Rightarrow$ Both firm cannot increase profits by deviating.

e) $c_1(q_1) = c_1 q_1$ $\qquad c_1 < c_2$
$c_2(q_2) = c_2 q_2$

$$q_1 = \begin{cases} 0 \text{ if } p_1 > p_2 \\ \alpha - \beta p_1 \text{ if } p_1 \leq p_2 \end{cases}$$

$$q_2 = \begin{cases} 0 \text{ if } p_2 \geq p_1 \\ \alpha - \beta p_2 \text{ if } p_2 < p_1 \end{cases}$$

A candidate for a NE is:

$$p_1^N = p_2^N = c_2 \Rightarrow \begin{cases} \pi_1^N > 0, \; q_1^N > 0 \\ \pi_2^N = 0, \; q_2^N = 0 \end{cases}$$

Assume that $c_2 < p^m$ (the monopoly price).

$\pi_2 < 0$ if $p_2 < p_1^N = c_2$.
$\pi_2 = 0$ if $p_2 > p_1^N$.

$\pi_1 = 0$ if $p_1 > p_2^N$.
Given that $p_1^N < p^m$, we have $\left.\frac{\partial \pi_1}{\partial p_1}\right|_{p_1^N} > 0$. Hence, firm 1 cannot increase profits with a lower price than $p_1^N = c_2$.

$\Rightarrow$ No firm can do better by unilateraly deviating from
$p_1^N = p_2^N = c_2$.

4.13)

Assumptions:

i) $p_1 = 20 + \frac{p_2}{2} - q_1 \Rightarrow q_1 = 20 + \frac{p_2}{2} - p_1$

$p_2 = 20 + \frac{p_1}{2} - q_2 \Rightarrow q_2 = 20 + \frac{p_1}{2} - p_2$

ii) $c_i(q_i) = 20 q_i$

iii) Price competition (Not Cournot.)

Solution:

Problem of firm 1:

$$\max_{p_1} V_1 = \left(20 + \frac{p_2}{2} - p_1\right)(p_1 - 20)$$

$$\frac{\partial V_1}{\partial p_1} = 20 + \frac{p_2}{2} - p_1 - p_1 + 20 = 0$$

$$\Rightarrow p_1^* = \frac{40 + p_2/2}{2} = \boxed{20 + \frac{p_2}{4} = p_1(p_2)}$$

Reaction fctn of firm 1.

Analogously, we have:

$$\boxed{p_2(p_1) = 20 + \frac{p_1}{4}}$$

Reaction function of firm 2.

NE: $p_2^N = p_2(p_1(p_2^N))$

$$\Rightarrow p_2^N = 20 + \frac{1}{4}\left(20 + \frac{p_2^N}{4}\right)$$

$$\Rightarrow \boxed{p_2^N = \frac{80}{3}}$$

By symmetry, we have

$$\boxed{p_1^N = \frac{80}{3}}$$

$$\Rightarrow q_1^N = 20 + \frac{40}{3} - \frac{80}{3} \Rightarrow \boxed{q_1^N = \frac{20}{3} = q_2^N}$$