J&R 4.14)

Assumptions:

- identical firms with $c(q_i) = q_i^2 + 1$.
- $p_i = 10 - 15 q_i - \sum_{\substack{j=1 \\ j \neq i}}^{J} q_j$

REMARK: It is NOT a good idea to assume from the outset that firms produce the same output levels. This should be derived from the NE conditions.

a) Short-run NE:

problem of firm i:

$$\max_{q_i} \pi_i = p_i q_i - c(q_i)$$

FOC $\frac{\partial \pi_i}{\partial q_i} = 10 - 15 q_i^* - \sum_{j \neq i} q_j - 15 q_i^* - 2 q_i^* = 0.$

$$\Rightarrow \boxed{q_i^* = \frac{10 - \sum_{j \neq i} q_j}{32}}$$ Reaction fctn of firm i.

Assuming a symmetrical NE, we have:

$$q_1^* = q_2^* = \ldots = q_J^* \equiv q^*$$

$$\Rightarrow q^* = \frac{10 - (J-1) q^*}{32} \Rightarrow \boxed{q^* = \frac{10}{J+31}}$$ NE output per firm

$$\Rightarrow p_i^* = 10 - 15\left(\frac{10}{J+31}\right) - (J-1)\left(\frac{10}{J+31}\right)$$

$$\boxed{p_i^* = 10 - \frac{10}{J+31}(J+14)}$$ NE price per firm.

$$\Rightarrow \Pi_i^* = \left(10 - 10\left(\frac{J+14}{J+31}\right)\right)\left(\frac{10}{J+31}\right) - \left(\frac{10}{J+31}\right)^2 - 1$$

$$= \left(\frac{10}{J+31}\right)^2 \left(J+31 - (J+14) - 1\right) - 1$$

$$\boxed{\Pi_i^* = 16\left(\frac{10}{J+31}\right)^2 - 1}$$ NE profit per firm.

b) Long-run NE:

$$\Pi_i^* = 0 \Rightarrow 16\left(\frac{10}{J+31}\right)^2 = 1$$

$$\Rightarrow \frac{10}{J+31} = \frac{1}{4}$$

$$\Rightarrow \boxed{J = 9}$$ Long-run NE numbers of active firms.

J-R. 4.16)

$\bar{u}(\vec{p}, m; q) = \max_{\vec{x}} u(q, \vec{x})$ s.t. $\vec{p}\vec{x} \le m$. [*]

a) Recall the definition of an indirect utility fctn (p 27):

$v(\vec{p}, y) = \max_{\vec{x}} u(\vec{x})$ s.t. $\vec{p}\vec{x} \le y$.

For fixed q, m and $\vec{p}$, the problem in [*] is exactly the same as the definition of an indirect utility fctn. Hence, $\bar{u}(\vec{p}, m; q)$ must have the same properties as $v(\vec{p}, y)$, with vector $(\vec{p}, y)$ being replaced by $(\vec{p}, m)$.

Since $v(\vec{p}, y)$ is strictly increasing in y, $\bar{u}(m, q)$ must be strictly increasing in m. ✓

And since $u(q, \vec{x})$ is strictly increasing in q, $\bar{u}(q, m)$ must be strictly increasing in q. ✓

$\bar{u}(q, m)$ is continuous by the theorem of the maximum (see p. 505).

We now want to show that $\bar{u}(q, m)$ is strictly quasi-concave.

We shall make use of the fact that $\bar{u}(q, m)$ is strictly quasi-concave iff its superior set is strictly convex; that is, we must show that

If $\bar{u}(q^1, m^1) \geq \alpha$ and $\bar{u}(q^2, m^2) \geq \alpha$, then $\bar{u}(q^t, m^t) > \alpha$, where

$$q^t = tq^1 + (1-t)q^2, \quad m^t = tm^1 + (1-t)m^2,$$
$$\forall t \in (0, 1).$$

<u>PROOF</u>:

Let $\vec{x}(m)$ be the solution to the problem in [*]. We have:

$$\bar{u}(q^1, m^1) = u(q^1, \vec{x}(m^1)) \geq \alpha$$

$$\bar{u}(q^2, m^2) = u(q^2, \vec{x}(m^2)) \geq \alpha$$

We must show that
$$\bar{u}(q^t, m^t) = u(q^t, \vec{x}(m^t)) > \alpha,$$
where $q^t = tq^1 + (1-t)q^2$
$\quad m^t = tm^1 + (1-t)m^2$

Since $u(q, x)$ is strictly quasi-concave, we have
$u(q^t, \vec{x}^t) > \alpha$, where $\vec{x}^t = t\vec{x}(m^1) + (1-t)\vec{x}(m^2)$.

Moreover,
$$\vec{p}\,\vec{x}^t = t\,\vec{p}\,\vec{x}(m^1) + (1-t)\,\vec{p}\,\vec{x}(m^2)$$
$$= t\,m^1 + (1-t)\,m^2$$
$$= m^t$$

By definition, $u(q^t, \vec{x}(m^t)) \geq u(q^t, \vec{x}^t)$.

Hence, $\bar{u}(q^t, m^t) = u(q^t, \vec{x}(m^t)) > \alpha$. <u>QED</u>

b) Since $u(q, \vec{x})$ is continuous and strictly quasi-concave, the Marshallian demands can be derived as follows:

$$\mathcal{L} = u(q, \vec{x}) + \lambda(y - pq - \vec{p}\vec{x})$$

$$\mathcal{L}_q = \frac{\partial u(q^*, \vec{x}^*)}{\partial q} - \lambda^* p = 0 \qquad [A]$$

$$\mathcal{L}_{x_i} = \frac{\partial u(q^*, \vec{x}^*)}{\partial x_i} - \lambda^* p_i = 0, \quad i = 1, \dots, n \qquad [B]$$

$$pq^* + \vec{p}\vec{x}^* = y \qquad [C]$$

With [A], [B] & [C], we have $n+2$ equations and $n+2$ variables: λ, q, x_i. This completely defines the Marshallian demands.

By the definition of $\bar{u}(q, m)$, we have:

$$\mathcal{L}' = u(q, \vec{x}) + \lambda'(m - \vec{p}\vec{x}), \quad q \text{ given.}$$

$$\frac{\partial \mathcal{L}'}{\partial x_i} = \frac{\partial u(q, \vec{x}'^*)}{\partial x_i} - \lambda'^* p_i = 0, \quad i = 1, \dots, n \qquad [A']$$

$$\vec{p}\vec{x}'^* = m \qquad [B']$$

By the envelope condition, we thus have:

$$\frac{\partial \bar{u}}{\partial m} = \frac{\partial \mathcal{L}'(\vec{x}'^*, \lambda'^*)}{\partial m} = \lambda'^* = \frac{1}{p_i} \frac{\partial u(q, \vec{x}'^*)}{\partial x_i} \qquad [E1]$$

$$\frac{\partial \bar{u}}{\partial q} = \frac{\partial \mathcal{L}'(\vec{x}'^*, \lambda'^*)}{\partial q} = \frac{\partial u(q, \vec{x}'^*)}{\partial q} \qquad [E2]$$

Let us now solve for

$$\max_{q,m} \bar{u}(q,m) \text{ s.t. } pq+m \leq y. \qquad [*]$$

Since $\bar{u}(q,m)$ is continuous and strictly quasi-concave, we have

$$\mathcal{L}''(q,m) = \bar{u}(q,m) + \lambda''(y-pq-m)$$

$$\frac{\partial \mathcal{L}''}{\partial q} = \frac{\partial \bar{u}(q''^*, m^*)}{\partial q} - \lambda''^* p = 0 \qquad [A'']$$

$$\frac{\partial \mathcal{L}''}{\partial m} = \frac{\partial \bar{u}(q''^*, m^*)}{\partial m} - \lambda''^* = 0 \qquad [B'']$$

$$pq^* + m^* = y \qquad [C'']$$

Substituting [E1], [E2] and [B'], we have

$$\frac{\partial u(q''^*, x''^*)}{\partial q} - \lambda''^* p = 0$$

$$\frac{1}{p_i} \frac{\partial u(q''^*, x''^*)}{\partial x_i} - \lambda''^* = 0, \ \forall i$$

$$pq''^* + \vec{p}\vec{x}''^* = y$$

Those are exactly the same conditions as [A], [B], and [C] which define the Marshallian demands. Therefore, solving for problem [*] is equivalent to solving for

$$\max_{q,\vec{x}} u(q,\vec{x}) \text{ s.t. } pq + \vec{p}\vec{x} \leq y.$$

and by definition, we have

$$\bar{u}(q^*, m^*) \equiv u(q^*, \vec{x}^*) = v(p, \vec{p}, y).$$

QED

c) Given that $\vec{q}$ is fixed, one can thus use problem [*] to analyze the full welfare effect of a change in the price of one commodity. The problem then reduces to choosing between two commodities:

① The one whose price change is being analyzed.

② Total expenditure on all other goods.

JRR 4.19) $u(x, m) = \ln(x) + m$

a) Marshallian dmds:

$$\max_{x,m} u(x, m) \text{ s.t. } px + m \leq y.$$

$$\mathcal{L}(x, m) = \ln(x) + m + \lambda(y - px - m)$$

$$\left.\begin{aligned} \mathcal{L}_x &= \frac{1}{x^*} - p\lambda^* = 0 \Rightarrow \lambda^* = \frac{1}{px^*} \\ \mathcal{L}_m &= 1 - \lambda^* = 0 \Rightarrow \lambda^* = 1 \end{aligned}\right\} \Rightarrow \boxed{x^* = \frac{1}{p}}$$

$$px^* + m^* = y \Rightarrow \boxed{m^* = y - 1}$$

$$\left.\begin{aligned} x(\vec{p}, y) &= 1/p \\ m(\vec{p}, y) &= y - 1 \end{aligned}\right\} \text{ Marshallian dmds.}$$

b) Indirect utility:

$$v(p, y) = u(x^*, m^*)$$

$$\Rightarrow \boxed{v(p, y) = \ln\left(\frac{1}{p}\right) + y - 1}$$

c) Slutsky decomposition:

$$\underset{TE}{\frac{\partial x}{\partial p}} = \underset{SE}{\frac{\partial x^h}{\partial p}} - \underset{IE}{x \frac{\partial x}{\partial y}}$$

Since $\partial x / \partial y = 0$, we have no income effect. The total effect of an own-price change is thus

$$\frac{\partial x}{\partial p} = \frac{-1}{p^2} = \frac{\partial x^h}{\partial p}.$$

It is a purely substitution effect as expenditures on m are not affected.

d) Consumer surplus effect of price change:

$$\Delta \text{welfare} = u(x(p^1), m(p^1)) - u(x(p^0), m(p^0))$$

$$= \int_{p^0}^{p^1} \frac{du}{dp}\, dp = \int_{p^0}^{p^1} \Big[\frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial m}\underbrace{\frac{\partial m}{\partial p}}_{=0} \Big]\, dp$$

$$= \int_{p^0}^{p^1} \frac{\partial u}{\partial x}\frac{\partial x}{\partial p}\, dp$$

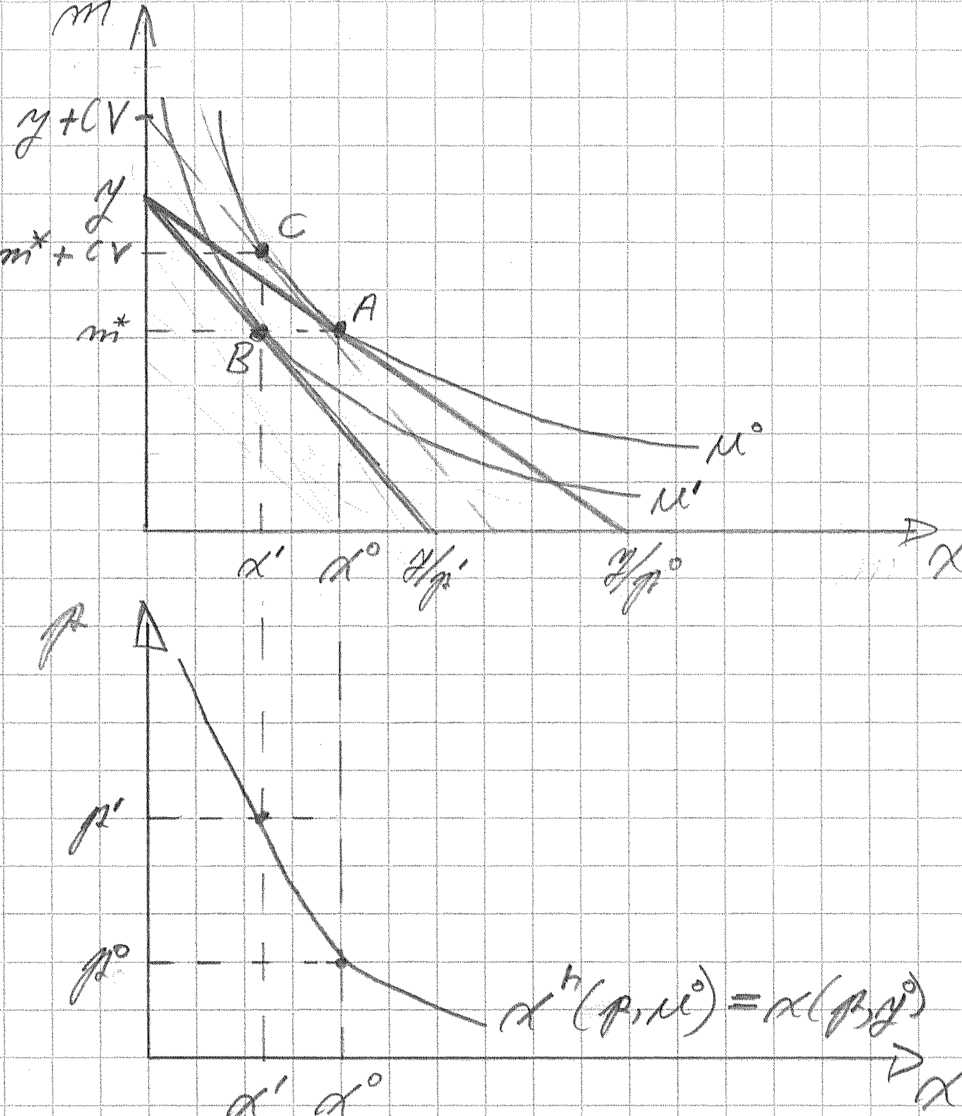
$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial p} = \frac{1}{x^*}\left(\frac{-1}{p^2}\right) = \frac{-1}{p} \quad \text{since } x^* = x(p) = \frac{1}{p}.$$

$$\Rightarrow \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} = -x(p)$$

$$\Rightarrow \Delta \text{welfare} = -\int_{p^0}^{p^1} x(p)\, dp \equiv \Delta CS \;. \quad \underline{QED}$$

by definition of CS.

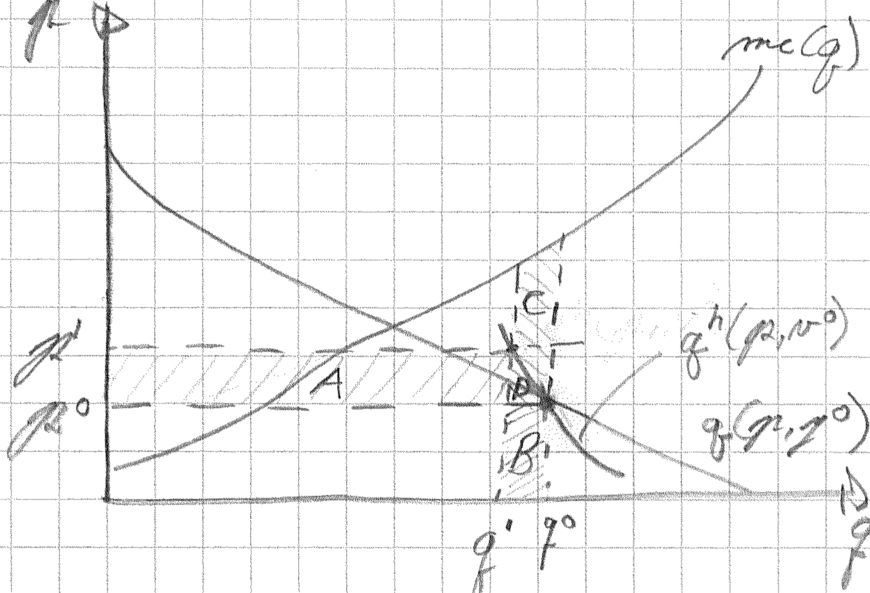
e)



NB Points A and B ensure that m^* does not vary with the price increase.

Points B and C ensure that x' does not vary with the increase in income.

J. & R. 4.21)



$p^0 \equiv$ initial price.

Consider an increase in price from p_0 to p_1.

The firm's change in profit is:

$$[p_1 q_1 - c(q_1)] - [p_0 q_0 - c(q_0)] = (p_1 q_1 - p_0 q_0) - (c(q_1) - c(q_0))$$

$$= (A - B) - \int_{q^0}^{q^1} mc(q)\,dq$$

$$= (A - B) + (B + C + D)$$

$$= A + C + D$$

Since $CV = A + D$, the consumer can be made indifferent while firm profits increase by C.

J.J.R 4.22)

$$p = \alpha - \beta q \qquad \alpha > c$$
$$C = cq + F \qquad (\alpha - c)^2 > 4\beta F.$$

a) Problem of monopoly:

$$\max_q \Pi = p(q)q - C(q) = (\alpha - \beta q)q - (cq + F)$$

$$\frac{\partial \Pi}{\partial q} = \alpha - 2\beta q^m - c = 0.$$

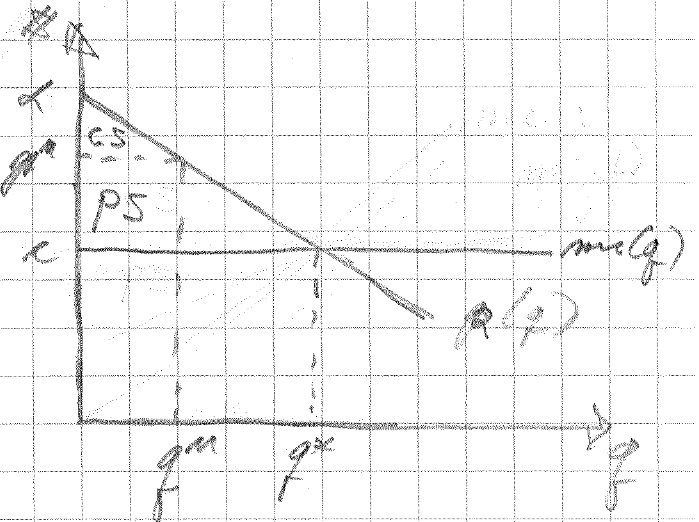
$$\Rightarrow \boxed{q^m = \frac{\alpha - c}{2\beta}}$$ monopoly output

$$p^m = \alpha - \beta q^m = \alpha - \frac{\alpha - c}{2} \Rightarrow \boxed{p^m = \frac{\alpha + c}{2}}$$ monopoly price

$$\Pi^m = \left(\frac{\alpha + c}{2} - c\right)\left(\frac{\alpha - c}{2\beta}\right) - F = \boxed{\frac{1}{\beta}\left(\frac{\alpha - c}{2}\right)^2 - F}$$ monopoly profit

b) Deadweight Loss:

Consumer surplus is max. at $p = mc$.



$$\Rightarrow p^* = \alpha - \beta q^* = c$$

$$\Rightarrow q^* = \frac{\alpha - c}{\beta} > q^m$$

$$\Rightarrow CS^* = \frac{(\alpha - c)}{2} q^* = \frac{(\alpha - c)^2}{2\beta}$$

N.B. $PS^* = 0$.

In the monopoly case, we have:

$$CS^m + PS^m = \frac{(\alpha - p^m)}{2} q^m + (p^m - c) q^m$$

$$= \left[\frac{\alpha}{2} - \left(\frac{\alpha+c}{4}\right) + \left(\frac{\alpha+c}{2} - c\right)\right]\left(\frac{\alpha-c}{2\beta}\right)$$

$$= \frac{1}{4}\left(2\alpha - \alpha - c + 2\alpha + 2c - 4c\right)\left(\frac{\alpha-c}{2\beta}\right)$$

$$= \left(\frac{\alpha-c}{8\beta}\right)\left(3\alpha - 3c\right)$$

$$\Rightarrow TS^m = \frac{3}{8\beta}(\alpha-c)^2 = \frac{3}{4}\left(\frac{(\alpha-c)^2}{2\beta}\right) = \frac{3}{4}\, CS^*$$

Hence $TS^m < TS^*$ by 25%.

c) At $p = c$, we have:

$\pi = -F$.

In order to get the firm to produce at max. TS, the firm will make losses. Hence, the government would have to subsidize this firm.

REMARK: Given that the fixed cost uses up real resources, it may be better not to produce at all.

J.&R. 4.27)

Assumptions:

- monopoly
- $q = p^{-\epsilon}$, $\epsilon > 1$. $\Rightarrow p(q) = q^{-1/\epsilon}$
- $tc(q) = cq$
- unit tax $\Rightarrow t(q) = tq$: total tax

Monopoly equil. $\Rightarrow mr(q) = mc(q)$

$tr(q) = p(q)q = q^{-1/\epsilon} \cdot q = q^{1-1/\epsilon}$

$\Rightarrow mr(q) = (1-\frac{1}{\epsilon})q^{-1/\epsilon}$ = marginal revenue.

With a tax t per unit of output, the cost of producing q can be expressed as: $(c+t)q$

The effective marginal cost is thus $t+c$. We thus have:

$mr = mc \Rightarrow (1-1/\epsilon)q_M^{-1/\epsilon} = c+t$

$$\Rightarrow q_M^{-1/\epsilon} = \frac{c+t}{1-1/\epsilon}$$

Since $p = q^{-1/\epsilon}$, we have:

$p_m(t) = \frac{c+t}{1-1/\epsilon}$ = monopoly price.

If $t=0$, we have $p_m(0) = \frac{c}{1-1/\epsilon}$.

Hence, with a unit tax t, the monopoly price increases by

$\frac{t}{1-1/\epsilon} > t$. QED.

J. & R. 4.29)

assumptions:
- Two periods, $t = 1, 2$.
- Monopoly
- $p_t = p(q_t)$
- r = discount rate.

a) $c_t = c(q_t)$:

problem of firm is to max. present discounted value (PDV):

$$\max_{q_1, q_2} PDV = p(q_1)q_1 - c(q_1) + \left(\frac{1}{1+r}\right)\left[p(q_2)q_2 - c(q_2)\right]$$

FOCs:

$$\frac{\partial PDV}{\partial q_1} = mr(q_1^*) - mc(q_1^*) = 0 \Rightarrow mr(q_1^*) = mc(q_1^*)$$

$$\frac{\partial PDV}{\partial q_2} = \left(\frac{1}{1+r}\right)\left[mr(q_2) - mc(q_2)\right] = 0$$

$$\Rightarrow mr(q_2^*) = mc(q_2^*).$$

Hence the firm maximizes period-by-period profits.

QED

b) Learning-by-doing:

$$c_1 = c_1(q_1)$$

$$c_2 = c_2(q_1, q_2), \quad \frac{\partial c_2}{\partial q_1} < 0.$$

The problem of the firm is:

$$\max_{q_1, q_2} PDV = p(q_1)q_1 - c_1(q_1) + \left(\frac{1}{1+r}\right)\left[p(q_2)q_2 - c_2(q_1, q_2)\right]$$

FOCs:

$$\frac{\partial PDV}{\partial q_1} = mr(q_1^*) - \frac{\partial c_1}{\partial q_1} - \left(\frac{1}{1+r}\right)\frac{\partial c_2}{\partial q_1} = 0. \qquad [1]$$

$$\frac{\partial PDV}{\partial q_2} = \left(\frac{1}{1+r}\right)\left[mr(q_2) - \frac{\partial c_2}{\partial q_2}\right] = 0 \qquad [2]$$

From [1], we have:

$$mr(q_1^*) = \frac{\partial c_1(q_1^*)}{\partial q_1} + \left(\frac{1}{1+r}\right)\frac{\partial c_2(q_1^*, q_2^*)}{\partial q_1} \qquad [1']$$

From [2], we have:

$$mr(q_2^*) = \frac{\partial c_2(q_1^*, q_2^*)}{\partial q_2} \qquad [2']$$

Since $\partial c_2/\partial q_1 < 0$, condition [1'] implies $mr(q_1^*) < \frac{\partial c_1(q_1^*)}{\partial q_1}$.

In order to benefit from the learning-by-doing effect, the firm will produce extra units at a loss during period 1. This loss will be compensated by the lower production costs in period 2. Note that the benefit from lower cost in period 2 must be discounted at rate r.