J&R 3.1) a) Show that $\frac{\partial AP_i}{\partial x_i} \cdot \frac{x_i}{AP_i} = \mu_i - 1$

where $\mu_i = \frac{f_i x_i}{f} = \frac{MP_i}{AP_i}$ where $MP_i = f_i$

$AP_i = \frac{f}{x_i}$

We have

$$\frac{\partial AP_i}{\partial x_i} = \frac{f_i x_i - f}{x_i^2} = \frac{1}{x_i}(MP_i - AP_i)$$

$$\Rightarrow \frac{\partial AP_i}{\partial x_i} \cdot \frac{x_i}{AP_i} = \frac{1}{x_i}(MP_i - AP_i) \cdot \frac{x_i}{AP_i}$$

$$= \frac{MP_i}{AP_i} - 1 = \mu_i - 1 \quad \underline{QED}$$

b) Since $\frac{\partial AP_i}{\partial x_i} = \frac{1}{x_i}(MP_i - AP_i)$,

we have the desired result.

J.&R. 3.2)

$$\frac{\partial}{\partial x_1} AP_1 = \frac{\partial}{\partial x_1} \frac{f(\vec{x})}{x_1} = \frac{f_1 x_1 - f(\vec{x})}{x_1^2} = \frac{1}{x_1}\left(f_1 - \frac{f}{x_1}\right)$$

If $\frac{\partial}{\partial x_1} AP_1 > 0$ then $x_1 f_1 > f(\vec{x})$. [*]

By CRS, we have $f(tx_1, tx_2) = ty$

$\Rightarrow f_1 x_1 + f_2 x_2 = y = f(\vec{x})$.

$\Rightarrow f_2 x_2 = f(\vec{x}) - f_1 x_1$

$\Rightarrow f_2 x_2 < 0$ from [*].

$\Rightarrow f_2 < 0$ since $x_2 > 0$. QED

J&R 3.3) By homo. of deg. 1, we have

$$f(t\vec{x}) = ty, \quad \forall t.$$

$$\Rightarrow \frac{\partial}{\partial t} f(t\vec{x}) = \sum_{i=1}^{n} f_i(t\vec{x}) x_i = y = f(\vec{x})$$

Let $t=1$, and we have

$$\sum_{i=1}^{n} MP_i(\vec{x}) x_i = f(\vec{x}) \quad \underline{QED}.$$

J.&R. 3.19)

$c(w,y) = A w_1^\alpha w_2^\beta y.$

In order to be increasing in $\vec{w}$, we need:

$$\frac{\partial c}{\partial w_1} = \alpha A w_1^{\alpha-1} w_2^\beta y \geq 0 \Rightarrow \alpha \geq 0, \quad A > 0$$

And similarly, $\beta \geq 0$.

In order to be homo. of deg 1 in $\vec{w}$, we need:

$$A\, t^\alpha w_1^\alpha t^\beta w_2^\beta y = t A w_1^\alpha w_2^\beta y$$

$$\Rightarrow t^\alpha t^\beta = t$$

$$\Rightarrow \alpha + \beta = 1$$

In order to be concave, its Hessian must be negative semi-definite. This implies:

$$\frac{\partial^2 c}{\partial w_1^2} \leq 0 \Rightarrow \alpha(\alpha-1) A w_1 w_2^\beta y \leq 0.$$

$\Rightarrow \alpha \leq 1$: This is already respected by $\beta, \alpha \geq 0$ and $\alpha + \beta = 1$.

In fact, it can be verified that the Hessian is neg. definite by the sufficient conditions with respect to its i-th principal minors:

$$(-1)^i D_i(x) > 0, \quad i = 1, 2.$$

J.&B. 3.42)

Let $y^0 = y(p^0, \vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}})$ and $\vec{x}^0 = \vec{x}(p^0, \vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}})$ be the SR supply and input demands at $(p^0, \vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}})$.

$\Rightarrow \pi(p^0, \vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}}) = p^0 y^0 - \vec{w}^0 \vec{x}^0 - \vec{\vec{w}} \vec{\vec{x}}$.

Let $y' = y(tp^0, t\vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}})$ and $\vec{x}' = \vec{x}(tp^0, t\vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}})$.

$\Rightarrow \pi(tp^0, t\vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}}) = tp^0 y' - t\vec{w}^0 \vec{x}' - \vec{\vec{w}} \vec{\vec{x}}$.

If y^0 and $\vec{x}^0$ does not maximize profit at tp^0 and $t\vec{w}^0$, we have:

$tp^0 y' - t\vec{w}^0 \vec{x}' - \vec{\vec{w}} \vec{\vec{x}} > tp^0 y^0 - t\vec{w}^0 \vec{x}^0 - \vec{\vec{w}} \vec{\vec{x}}$

$\Rightarrow \quad p^0 y' - \vec{w}^0 \vec{x}' > p^0 y^0 - \vec{w}^0 \vec{x}^0$.

This implies that y^0 and $\vec{x}^0$ do not max. profits at $(p^0, \vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}})$.

a contradiction.

Hence, $\vec{y}^0 \vec{x}^0$ must maximize profits at $(tp^0, t\vec{w}^0, \vec{\vec{w}}, \vec{\vec{x}})$.

QED.