J.&R. 1.54) ①

$u(\vec{x}) = \sum_i f_i(x_i) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n)$,
with $\frac{\partial}{\partial x_i} f_i > 0$.

and $u(\vec{x})$ strictly quasi-concave.

a) Show that if, for some good j,
$f_j''(x_j^*) > 0$ then $f_i''(x_i^*) < 0 \ \forall i \neq j$.

PROOF: Take any two pairs of goods, say good i and good j, and let $p_i x_i^* + p_j x_j^* = \bar{y}$, i.e. $\bar{y}$ is the part of the budget spent on goods i and j optimally. If x_i^* and x_j^* maximize utility, it must solve:

$$\max_{x_i, x_j} u = f_i(x_i) + f_j(x_j) \text{ s.t. } p_i x_i + p_j x_j = \bar{y}.$$

Substituting, we get:

$$\max_{x_i} u = f_i(x_i) + f_2\left(\frac{\bar{y} - p_i x_i}{p_j}\right)$$

FOC: $\frac{\partial u}{\partial x_i} = f_i' - f_j' \cdot \frac{p_i}{p_j} = 0$

SOC: $\frac{\partial^2 u}{\partial x_i^2} = f_i'' + f_j'' \left(\frac{p_i}{p_j}\right)^2 < 0.$

If $f_i'' > 0$, then it must be the case that $f_j'' < 0$ for the second-order condition to be verified. QED

ALTERNATIVE PROOF: Suppose goods 1 and 2 both have increasing marginal utility. In an interior solution, we have:

$$f_1'(x_1^*) = \frac{p_1}{p_2} f_2'(x_2^*)$$

Let $\Delta x_1 = +1$ and $\Delta x_2 = -p_1/p_2$. Total expenditure is unaffected and we now have:

$$f_1'(x_1^* + \Delta x_1) > \frac{p_1}{p_2} f_2'(x_2^* - p_1/p_2)$$

since $f_1'' > 0$ and $f_2'' > 0$. Hence, it is better to keep on reducing x_2 in favor of x_1, i.e. $x_2 = 0$.

1.54)

b) Normal good i: $\frac{\partial x_i}{\partial y} > 0 \Rightarrow \frac{\partial x_i}{\partial p_i} < 0$.

Inferior good j: $\frac{\partial x_j}{\partial y} < 0$ and $\frac{\partial x_j}{\partial p_j} > 0 \Rightarrow \frac{\partial x_j}{\partial y}$.

Let $f_1'' > 0$ and $f_j'' < 0$, $j = 2, \ldots, n$.

2 steps: ① Show that i and j cannot be both normal.
② Show that i must be normal.

PROOF: Since $x_1^* > 0$ and $x_2^* > 0$, we initially have:

$$\frac{f_1'(x_1^*)}{f_2'(x_2^*)} = \frac{p_1}{p_2}.$$

① Assume that both are normal goods, and that y increases by Δy, we have:

$$\Delta x_1^* = \frac{\partial x_1}{\partial y}\Delta y > 0 \text{ and } \Delta x_2^* = \frac{\partial x_2}{\partial y}\Delta y > 0.$$

$\Rightarrow x_1' \equiv x_1^* + \Delta x_1^* > x_1^*$ and $x_2' \equiv x_2^* + \Delta x_2^* > x_2^*$ at $y' = y + \Delta y$.

$\Rightarrow f_1'(x_1') > f_1'(x_1^*)$ since $f_1'' > 0$

$f_2'(x_2') < f_2'(x_2^*)$ since $f_2'' < 0$.

$\Rightarrow \frac{f_1'(x_1')}{f_2'(x_2')} > \frac{p_1}{p_2}$. A contradiction.

Hence, good 1 and any of goods $j = 2, \ldots, n$ cannot be both normal.

We can similarly show that good 1 and any of goods $j = 2, \ldots, n$ cannot be both inferior.

1.54) b) <u>ALTERNATIVE PROOF</u>

① We want to show that if $f_1'' > 0$ and $f_2'' < 0$, then both cannot be normal goods:

In an interior solution for x_1^* and x_2^*, we have:

$$\psi \equiv p_2 f_1'(x_1^*) - p_1 f_2'(x_2^*) = 0.$$

Implicit differentiation yields:

$$\frac{\partial x_1^*}{\partial x_2^*} = -\frac{\psi_{x_2}}{\psi_{x_1}} = -\frac{p_2 f_1''(x_1^*)}{-p_1 f_2''(x_2^*)} < 0.$$

This says that if the price ratio remains constant, any increase in x_2^* must be accompanied by a decrease in x_1^*, or vice versa.

In particular, if good 1 is normal and thus increases with income, then good 2 must decrease with income.

<u>QED</u>

② It remains to be shown that good 1 is normal.

Assume not. Then, according to part a), goods $j = 2, \ldots, n$ are all normal. If y increases by Δy, we thus have:

$\Delta x_1^* < 0$ and $\Delta x_j > 0$, $\forall j = 2, \ldots, n$.

(I can't figure out this proof. Intuitively, I suspect it is wrong.)

c) $f_i'' < 0$, $i = 1, \ldots, n$.

Show that all goods are normal.

PROOF: First of all, all goods cannot be inferior at the same time lest the budget balancedness is violated.

It remains to be shown that one good cannot be inferior while another is normal. Assume it is the case that good 1 is normal while good 2 is inferior. Following an increase in income Δy, we have

$$\Delta x_1^* = \frac{\partial x}{\partial y} \Delta y > 0 \text{ and } \Delta x_2^* = \frac{\partial x_2}{\partial y} \Delta y < 0.$$

Let $x_1' = x_1^* + \Delta x_1^* > x_1^*$ and $x_2' = x_2^* + \Delta x_2^* > x_2^*$

where $\dfrac{f_1'(x^*)}{f_2'(x_2^*)} = \dfrac{p_1}{p_2}$.

$$\Rightarrow \frac{f_1'(x_1')}{f_2'(x_2')} < \frac{p_1}{p_2}. \text{ A contradiction.}$$

J. & R. 1.55)

a) $v(\vec{p}, y) = f(y)\, p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$.

Given the properties in THEOREM 1.6, we must have:

① $f(y)$ continuous in y. ✓

② Homo. of deg. 0 in $(\vec{p}, y)$ implies

$f(ty)(tp_1)^{\alpha_1}(tp_2)^{\alpha_2}(tp_3)^{\alpha_3} = f(y)\, p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$

$\Rightarrow f(ty)\, t^{\alpha_1+\alpha_2+\alpha_3} = f(y)$

$\Rightarrow f(ty) = t^{-(\alpha_1+\alpha_2+\alpha_3)} f(y)$

$\Rightarrow f(y)$ is homo. of deg. $-(\alpha_1+\alpha_2+\alpha_3)$. ✓

③ v strictly increasing in y implies

$f'(y) > 0$. ✓

④ v decreasing in $\vec{p}$ implies:

$\alpha_1, \alpha_2, \alpha_3 < 0$. ✓

⑤ Quasiconvex in $(\vec{p}, y)$ implies:

$v(\vec{p}, y)$ is quasi-convex iff $-v(\vec{p}, y)$ is quasi-concave.

A necessary condition for $-v(\vec{p}, y)$ to be quasi-concave is:

1.55) a) cont'd

$$(-1)^r D_r(\vec{p}, y) \geq 0$$

(Unfortunately, this is not given in Jehle & Reny.)

where $D_r = \begin{vmatrix} 0 & \frac{\partial \bar{v}}{\partial p_1} & \frac{\partial \bar{v}}{\partial p_2} & \frac{\partial \bar{v}}{\partial p_3} & \frac{\partial \bar{v}}{\partial y} \\ \frac{\partial \bar{v}}{\partial p_1} & \frac{\partial^2 \bar{v}}{\partial p_1^2} & \frac{\partial^2 \bar{v}}{\partial p_1 \partial p_2} & \cdots & \frac{\partial^2 \bar{v}}{\partial p_1 \partial y} \\ \frac{\partial \bar{v}}{\partial p_2} & \vdots & \ddots & & \\ \frac{\partial \bar{v}}{\partial p_3} & \vdots & & \ddots & \\ \frac{\partial \bar{v}}{\partial y} & & & & \frac{\partial^2 \bar{v}}{\partial y^2} \end{vmatrix}$

$\bar{v} = -v.$

$$(-1)^1 D_1 = \left(\frac{\partial \bar{v}}{\partial p_1}\right)^2 = \left(\frac{\alpha_1 v(\vec{p}, y)}{p_1}\right)^2 \geq 0$$ (not useful b/c always true.)

$$(-1)^2 D_2 = \begin{vmatrix} 0 & \bar{v}_1 & \bar{v}_2 \\ \bar{v}_1 & \bar{v}_{11} & \bar{v}_{12} \\ \bar{v}_2 & \bar{v}_{21} & \bar{v}_{22} \end{vmatrix} \geq 0$$

The above inequality must hold for $-v(\vec{p}, y)$ to be quasiconcave. It is a long procedure, but that's how I would do it...

1.55 b)

$v(p_1, p_2, y) = u(p_1, p_2) + z(p_1, p_2)/y$

① Continuous. ✓

② Homo. of deg. 0 in $(\vec{p}, y)$:

$u(tp_1, tp_2) + \frac{z(tp_1, tp_2)}{ty} = u(p_1, p_2) + \frac{z(p_1, p_2)}{y}$

$\Rightarrow u(p_1, p_2)$ is homo. of deg. 0 in p_1, p_2.

$z(p_1, p_2)$ is homo. of deg. 1 in p_1, p_2.

③ v strictly increasing in y implies:

$z(p_1, p_2) < 0$.

④ v decreasing in $\vec{p}$ implies:

$\frac{\partial u}{\partial p_i} < 0$, $i = 1, 2$. (Suppose $y \to \infty$.)

$\frac{\partial z}{\partial p_i} < 0$, $i = 1, 2$. (Suppose $y \to 0$.)

⑤ Quasiconvex in $(\vec{p}, y)$:

i.e. $-v(\vec{p}, y)$ quasi concave?

$-v = -u(p_1, p_2) - \frac{z(p_1, p_2)}{y}$

2'

A necessary condition:

$$(-1)^n D_n \geq 0$$

$$D_n = \begin{vmatrix} 0 & -\left(\mu_1 + \frac{Z_1}{y}\right) & -\left(\mu_2 + \frac{Z_2}{y}\right) & \frac{Z}{y^2} \\ -\left(\mu_1 + \frac{Z_1}{y}\right) & -\left(\mu_{11} + \frac{Z_{11}}{y}\right) & -\left(\mu_{12} + \frac{Z_{12}}{y}\right) & \frac{Z_1}{y^2} \\ -\left(\mu_2 + \frac{Z_2}{y}\right) & \cdot & \cdot & \vdots \\ \frac{Z}{y^2} & \cdot & \cdot & \frac{-2Z}{y^3} \end{vmatrix}$$

WOW!

J & R. 1.61)

Hicks' 3rd Law:

$$\sum_{j=1}^{n} \frac{\partial x_i^h(\vec{p}, u)}{\partial p_j} p_j = 0$$

a) PROOF: Since $\vec{p}\,\vec{x}(\vec{p}, y) = y$, we have

$$\sum_{i=1}^{n}\left[p_i \frac{\partial x_i}{\partial p_j}\right] + x_j = 0$$

Using the Slutsky decomposition:

$$\sum_{i=1}^{n}\left[p_i \frac{\partial x_i^h}{\partial p_j} - x_j\, p_i \frac{\partial x_i}{\partial y}\right] + x_j = 0. \qquad [*]$$

Again, since $\vec{p}\,\vec{x}(\vec{p}, y) = y$, we have

$$\sum_i p_i \frac{\partial x_i}{\partial y} = 1.$$

Inserting into [*], we get:

$$\sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial p_j} = 0. \qquad \underline{QED}$$

b) Verify with

$$u(\vec{x}) = A\prod_{i=1}^{n} x_i^{\alpha_i}.$$

We had found that

$$x_j^h = \frac{u}{A}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\frac{\alpha_j}{p_j}$$

$$\Rightarrow \frac{\partial x_j^h}{\partial p_k} = \frac{u}{A}\frac{\alpha_k}{p_k}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\frac{\alpha_j}{p_j}\ ,\quad k\neq j$$

$$\frac{\partial x_j^h}{\partial p_j} = \frac{u}{A}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\left[\left(\frac{\alpha_j}{p_j}\right)^2 - \frac{\alpha_j}{p_j^2}\right]$$

$$\Rightarrow \frac{\partial x_j^h}{\partial p_k}p_k = \frac{u}{A}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\frac{\alpha_k\alpha_j}{p_j}$$

$$\frac{\partial x_j^h}{\partial p_j}p_j = \frac{u}{A}\left[\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\left(\frac{\alpha_j^2-\alpha_j}{p_j}\right)$$

$$\Rightarrow \sum_{j=1}^{n}\frac{\partial x_i^h}{\partial p_j}p_j = \frac{u}{A}\prod_i\left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\left\{\underbrace{\sum_{j\neq i}\left[\frac{\alpha_j\alpha_i}{p_i}\right] + \frac{\alpha_i^2-\alpha_i}{p_i}}\right\}$$

$$= \frac{\alpha_i}{p_i}\sum_{j\neq i}\alpha_j + \frac{\alpha_i^2}{p_i} - \frac{\alpha_i}{p_i}$$

$$= \frac{\alpha_i}{p_i}\sum_{j=1}^{n}\alpha_j - \frac{\alpha_i}{p_i} = 0.$$

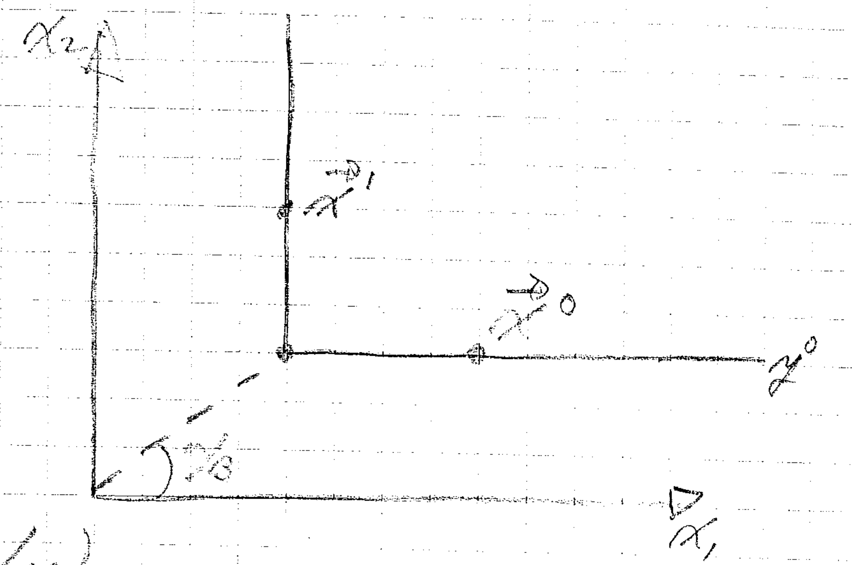
J. & R. #3.8)

$y = \min\{\alpha x_1, \beta x_2\}$, $\alpha, \beta > 0$.

$\alpha x_1 = \beta x_2 \Rightarrow y = \alpha x_1 = \beta x_2$

$\Rightarrow x_2 = \frac{\alpha}{\beta} x_1$

$\Rightarrow \alpha x_1 \geq \beta x_2 \Rightarrow y = \beta x_2$



$$\sigma_{12} = \frac{d \ln(x_2/x_1)}{d \ln(f_1/f_2)}$$

At $\vec{x}^0$: $f_1 = 0$ and $f_2 = \beta$.

J. & R. #3.9)

$$y = A x_1^{\alpha} x_2^{\beta}, \quad A, \alpha, \beta > 0.$$

(Follow procedure in example 3.1)

$$d\ln(x_2/x_1) = d\ln x_2 - d\ln x_1 = \frac{1}{x_2}dx_2 - \frac{1}{x_1}dx_1$$

$$\ln\left(\frac{f_1}{f_2}\right) = \ln\left(\frac{A\alpha x_1^{\alpha-1} x_2^{\beta}}{A\beta x_1^{\alpha} x_2^{\beta-1}}\right) = \ln\left(\frac{\alpha}{\beta}\,\frac{x_2}{x_1}\right)$$

$$= \ln \alpha/\beta + \ln x_2 - \ln x_1$$

$$\Rightarrow d\ln\left(\frac{f_1}{f_2}\right) = \frac{1}{x_2}dx_2 - \frac{1}{x_1}dx_1$$

$$\Rightarrow \sigma = \frac{d\ln(x_2/x_1)}{d\ln(f_1/f_2)} = 1$$