J&R 1.37) The expenditure fctn obtained from a CES utility fctn is

$$e(\vec{p}, u) = u(p_1^r + p_2^r)^{1/r}$$

where $r = \rho/(\rho-1)$, $0 \neq \rho < 1$.

Theorem 1.7 says that $e(\vec{p}, u)$ must have the following properties:

① $e(\vec{p}, u) = 0$ when $u(\vec{x})$ is lowest?

Since $u(\vec{x}) = (x_1^\rho + x_2^\rho)^{1/\rho}$, the lowest value is when $\vec{x} = \vec{0}$, which yields $u = 0$. Substituting into $e(\vec{p}, u)$ above, we do get $e(\vec{p}, 0) = 0$. ✓

② Continuous in $R_{++}^n \times U$?

$u(p_1^r + p_2^r)^{1/r}$ is continuous in u, p_1, and p_2 ✓

③ The function is increasing and unbounded above in u, $\forall \vec{p} \gg 0$?

$$\frac{\partial e}{\partial u} = (p_1^r + p_2^r)^{1/r} > 0 \Rightarrow \text{increasing} \checkmark$$

$$\lim_{u \to \infty} e = +\infty. \Rightarrow \text{unbounded above} \checkmark$$

④ $e(\vec{p}, u)$ is increasing in $\vec{p}$?

Is $u(p_1^{0r} + p_2^{0r})^{1/r} \geq u(p_1^r + p_2^r)^{1/r}$, $\forall \vec{p}^0 \geq \vec{p}$?

Whether $r>0$ or $r<0$, we have the following:

$$\left(p_1^{0r}+p_2^{0r}\right)^{1/r} \geq \left(p_1^{1r}+p_2^{1r}\right)^{1/r} \quad \forall\, \vec{p}^{\,0} \geq \vec{p}^{\,1}. \checkmark$$

⑤ Homogeneous of deg. 1 in $\vec{p}$?

$$e(t\vec{p},u) = u\left((tp_1)^r+(tp_2)^r\right)^{1/r} = u\left[t^r\left(p_1^r+p_2^r\right)\right]^{1/r}$$
$$= t\,u\left(p_1^r+p_2^r\right)^{1/r} = t\,e(\vec{p},u). \checkmark$$

⑥ Concave in $\vec{p}$?

Is the Hessian matrix of $e(\vec{p})$ negative semi-definite?

$1-r+r-1=0.$

We have:

$$\frac{\partial e}{\partial p_1} = u\cdot\frac{1}{r}\left(p_1^r+p_2^r\right)^{\frac{1}{r}-1}\cdot r p_1^{r-1} = \frac{e(\vec{p},u)}{p_1^{1-r}\left(p_1^r+p_2^r\right)} > 0$$

$$\frac{\partial^2 e}{\partial p_1^2} = \frac{1}{p_1^{1-r}\left(p_1^r+p_2^r\right)}\frac{\partial e}{\partial p_1} - \frac{e(\vec{p},u)}{\left(p_1^{1-r}\left(p_1^r+p_2^r\right)\right)^2}\cdot\left[(1-r)p_1^{-r}\left(p_1^r+p_2^r\right)+r\right]$$

$$= \frac{e(\vec{p},u)}{\left(p_1^{1-r}\left(p_1^r+p_2^r\right)\right)^2}\,(1-r)\left(1-\frac{p_1^r+p_2^r}{p_1^r}\right) < 0$$

$\checkmark$

It remains to be verified that

$$\begin{vmatrix} \dfrac{\partial^2 e}{\partial p_1^2} & \dfrac{\partial^2 e}{\partial p_1\partial p_2} \\ \dfrac{\partial^2 e}{\partial p_2\partial p_1} & \dfrac{\partial^2 e}{\partial p_2^2} \end{vmatrix} > 0$$

...

J&R

1.39) $x_i(\vec{p}, y)$ is homo. of deg. 0 in $(\vec{p}, y)$.

Roy's identity: $x_i = -\dfrac{\partial v / \partial p_i}{\partial v / \partial y}$

Theorem A2.6: If $f(\vec{x})$ is homo of deg. k, then $\dfrac{\partial}{\partial x_i} f(t\vec{x}) = t^{k-1} \dfrac{\partial f(\vec{x})}{\partial x_i}$, $\forall i$.

PROOF: Since $v(\vec{p}, y)$ is homo. of deg. 0 in $(\vec{p}, y)$, then $\dfrac{\partial}{\partial p_i} v(t\vec{p}, ty) = t^{-1} \dfrac{\partial}{\partial p_i} v(\vec{p}, y)$

$$\frac{\partial}{\partial y} v(t\vec{p}, ty) = t^{-1} \frac{\partial}{\partial y} v(\vec{p}, y)$$

$$\Rightarrow x_i(t\vec{p}, ty) = -\frac{\frac{\partial}{\partial p_i} v(t\vec{p}, ty)}{\frac{\partial}{\partial y} v(t\vec{p}, ty)} = -\frac{t^{-1} \frac{\partial}{\partial p_i} v(\vec{p}, y)}{t^{-1} \frac{\partial}{\partial y} v(\vec{p}, y)}$$

$$= x_i(\vec{p}, y) \quad \underline{QED}$$

J&R 1.41)

Def. of a normal good i: $\Delta^+ y \Rightarrow \Delta^+ x_i$.

① Show that if $\Delta^+ y \Rightarrow \Delta^+ x_i$ then $\Delta^- p_i \Rightarrow \Delta^+ x_i$.

PROOF: From the Slutsky eqtn, we have:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - x_i \frac{\partial x_i}{\partial y} \quad [*]$$

We also know that the own-substitution effect must be negative; i.e.

$$\frac{\partial x_i^h}{\partial p_i} \leq 0.$$

Hence, if $\frac{\partial x_i}{\partial y} > 0$, the RHS of [*] must be negative. QED

② Show that if $\Delta^+ p_i \Rightarrow \Delta^+ x_i$ then $\Delta^+ y \Rightarrow \Delta^- x_i$.

PROOF: Since $\frac{\partial x_i^h}{\partial p_i} \leq 0$, the Slutsky eqtn [*] tells that with $\frac{\partial x_i}{\partial p_i} > 0$, it must be the case that $\frac{\partial x_i}{\partial y} < 0$. QED

③ Show that if $\Delta^- p_i \Rightarrow \Delta^+ x_i$ then it is not necessarily the case that $\Delta^+ y \Rightarrow \Delta^+ x_i$.

Let $\frac{\partial x_i}{\partial y} < 0$ while $x_i \frac{\partial x_i}{\partial y} < -\frac{\partial x_i^h}{\partial p_i}$. Then

$$\frac{\partial x_i}{\partial p_i} < 0. \qquad \text{QED}$$

4 Show that if $\Delta^+ y \Rightarrow \Delta^- x_i$ it is not necessarily the case that $\Delta^+ p_i \Rightarrow \Delta^+ x_i$.

Assume that $\frac{\partial x_i}{\partial y} < 0$ while

$-x_i \frac{\partial x_i}{\partial y} < -\frac{\partial x_i^h}{\partial p_i}$. Then, according to the Slutsky eqtn, we have

$\frac{\partial x_i}{\partial p_i} < 0$. QED

J. & R. 1.53)

$$u(\vec{x}) = A\prod_{i=1}^{n} x_i^{\alpha_i}, \quad A>0, \quad \sum_{i=1}^{n}\alpha_i = 1.$$

$$\Rightarrow u(\vec{x}) = A x_1^{\alpha_1} x_2^{\alpha_2}\cdots x_n^{\alpha_n}.$$

a) Marshallian demand:

$$\max_{\{x_i\}} u(\vec{x}) \quad \text{s.t.} \quad \vec{p}\vec{x} = y.$$

$$\mathcal{L} = u(\vec{x}) + \lambda[y - \vec{p}\vec{x}]$$

$$\mathcal{L}_j = \frac{\partial u}{\partial x_j} - \lambda p_j = 0, \quad \forall j = 1, \ldots, n.$$

where $\dfrac{\partial u}{\partial x_j} = \dfrac{\alpha_j A\prod_{i=1}^{n} x_i^{\alpha_i}}{x_j}$

Note that $\lim_{x_j \to 0} \dfrac{\partial u}{\partial x_j} = +\infty$. Hence, $x_j^* > 0, \forall j$.

$$\Rightarrow \frac{\alpha_j u(\vec{x}^*)}{x_j^*} = \lambda p_j, \quad \forall j.$$

$$\Rightarrow \frac{\dfrac{\alpha_j u(\vec{x}^*)}{x_j^*}}{\dfrac{\alpha_k u(\vec{x}^*)}{x_k^*}} = \frac{p_j}{p_k}, \quad \forall j, k.$$

$$\Rightarrow \frac{\alpha_j x_k^*}{\alpha_k x_j^*} = \frac{p_j}{p_k} \Rightarrow x_j^* = \left(\frac{\alpha_j}{\alpha_k}\right)\left(\frac{p_k}{p_j}\right) x_k^*.$$

Inserting in the budget constraint:

$$p_1 x_1^* + \frac{\alpha_2}{\alpha_1} \cdot p_1 x_1^* + \frac{\alpha_3}{\alpha_1} p_1 x_1^* + \dots + \frac{\alpha_n}{\alpha_1} p_1 x_1^* = y$$

$$\Rightarrow x_1^* = \frac{y}{\frac{p_1}{\alpha_1} \sum_{i=1}^{n} \alpha_i} = \frac{\alpha_1 y}{p_1}$$

And similarly for all x_j^*:

$$x_j(\vec{p}, y) = \alpha_j \frac{y}{p_j}$$: This is the marshallian demand.

b) Indirect utility fctn:

$$v(\vec{p}, y) = u(\vec{x}(\vec{p}, y))$$

$$= A\left(\frac{\alpha_1 y}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2 y}{p_2}\right)^{\alpha_2} \cdots \left(\frac{\alpha_n y}{p_n}\right)^{\alpha_n}$$

$$= A\, y \prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}$$: This is the indirect utility fctn.

③

c) The expenditure fctn:

$$v(\vec{p}, e(\vec{p},u)) = u$$

$$\Rightarrow \quad A\, e(\vec{p},u) \; \Pi_i \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} = u$$

$$\Rightarrow e(\vec{p},u) = \frac{u}{A} \; \Pi_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}$$

d) The Hicksian demand:

By Shephard's Lemma, we have

$$x_j^h(\vec{p},u) = \frac{\partial e}{\partial p_j}$$

$$= \frac{u}{A}\left[\Pi_i \left(\frac{p_i}{\alpha_i}\right)^{\alpha_i}\right]\frac{\alpha_j}{p_j}$$