J. & R #1.57) STONE-GEARY UTILITY FCTN.

$$u(\vec{x}) = \prod_{i=1}^{n} (x_i - a_i)^{b_i}, \quad a_i, b_i \geq 0, \quad \sum_{i=1}^{n} b_i = 1$$

a) i) INDIRECT UTILITY FCTN:

$$\max_{\vec{x}} \prod_{i=1}^{n} (x_i - a_i)^{b_i} \quad \text{s.t.} \quad \vec{p}\,\vec{x} \leq y$$

$$\mathcal{L} = \prod_{i=1}^{n} (x_i - a_i)^{b_i} + \lambda\left(y - \sum_{i=1}^{n} p_i x_i\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \prod_{i=1}^{n} (x_i - a_i)^{b_i} \frac{b_j}{x_j - a_j} - \lambda p_j = 0, \quad j = 1, \ldots, n$$

$$\Rightarrow \frac{b_j}{(x_j - a_j) p_j} = \frac{\lambda}{u(\vec{x})}, \quad \forall j = 1, \ldots, n$$

$$\Rightarrow \frac{b_j}{(x_j - a_j) p_j} = \frac{b_i}{(x_i - a_i) p_i}, \quad \forall i, j$$

$$\Rightarrow x_j - a_j = (x_1 - a_1) \frac{p_1}{p_j} \frac{b_j}{b_1}, \quad \forall j \qquad \boxed{*}$$

$$\Rightarrow p_j x_j^* = (x_1^* - a_1) p_1 \frac{b_j}{b_1} + p_j a_j$$

$$\Rightarrow \sum_{j=1}^{n} p_j x_j = \sum_{j=1}^{n} (x_1^* - a_1) p_1 \frac{b_j}{b_1} + p_j a_j$$

$$= \frac{p_1}{b_1}(x_1^* - a_1) + \sum_{j=1}^{n} p_j a_j = y$$

$$\Rightarrow x_1^* - a_1 = \left(y - \sum p_j a_j\right) \frac{b_1}{p_1}$$

from [*]

$$\Rightarrow v(\vec{p}, y) = \prod_{i=1}^{n}\left[(x_i^* - a_i)\frac{p_i}{p_i}\frac{b_i}{b_i}\right]^{b_i}$$

$$v(\vec{p}, y) = \prod_{j=1}^{n}\left[\left(y - \sum p_j a_j\right)\frac{b_j}{p_j}\right]^{b_j} = \boxed{\left(y - \sum p_j a_j\right)\prod_{j=1}^{n}\left(\frac{b_j}{p_j}\right)^{b_j}}$$

$\Rightarrow$ proportional to $\underbrace{y - \sum p_j a_j}_{\text{discretionary income}}$

ii) <u>EXPENDITURE FCTN:</u>

$$u = v(\vec{p}, e(\vec{p}, u))$$

$$\Rightarrow u = \left(e(\vec{p}, u) - \sum p_j a_j\right)\prod\left(\frac{b_j}{p_j}\right)^{b_j}$$

$$\Rightarrow \boxed{e(\vec{p}, u) = \frac{u}{\prod\left(\frac{b_j}{p_j}\right)^{b_j}} + \sum p_j a_j} \Rightarrow \text{linear in } u$$

b) By Roy's identity, we have

$$x_i(\vec{p}, y) = -\frac{\partial v / \partial p_i}{\partial v / \partial y}$$

where $\dfrac{\partial v}{\partial p_i} = -a_i \prod_{j=1}^{n}\left(\dfrac{b_j}{p_j}\right)^{b_j} - \dfrac{b_i}{p_i}\left(y - \sum p_j a_j\right)\prod_{j=1}^{n}\left(\dfrac{b_j}{p_j}\right)^{b_j}$

$$\frac{\partial v}{\partial y} = \prod_{j=1}^{n}\left(\frac{b_j}{p_j}\right)^{b_j}$$

$$\Rightarrow x_i(\vec{p}, y) = a_i + \frac{b_i}{p_i}\left(y - \sum p_j a_j\right)$$

$$\Rightarrow \underbrace{p_i x_i(\vec{p}, y) = p_i \gamma_i}_{\text{Subsistence expenditure on good } i} + \underbrace{b_i \left(y - \sum p_j \gamma_j\right)}_{\text{Expenditure on good } i \text{ in excess of subsistence expenditure}}$$