

Q. & R. 1.55)

$$u(\vec{x}) = \sum_i f_i(x_i) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n)$$

with $\frac{\partial}{\partial x_i} f_i > 0$.

and $u(\vec{x})$ strictly quasi-concave.

a) Show that if, for some good j ,
 $f_j''(x_j^*) > 0$ then $f_i''(x_i^*) < 0 \forall i \neq j$.

PROOF: Take any two pairs of goods, say good i and good j , and let $p_i x_i^* + p_j x_j^* = \bar{y}$, i.e. $\bar{y}$ is the part of the budget spent on goods i and j optimally. If x_i^* and x_j^* maximizes utility, it must solve:

$$\max_{x_i, x_j} u = f_i(x_i) + f_j(x_j) \text{ s.t. } p_i x_i + p_j x_j = \bar{y}.$$

Substituting, we get:

$$\max_{x_i} u = f_i(x_i) + f_j\left(\frac{\bar{y} - p_i x_i}{p_j}\right)$$

$$\text{FOC: } \frac{\partial u}{\partial x_i} = f_i' - f_j' \cdot \frac{p_i}{p_j} = 0$$

$$\text{SOC: } \frac{\partial^2 u}{\partial x_i^2} = f_i'' + f_j'' \left(\frac{p_i}{p_j}\right)^2 < 0.$$

If $f_i'' > 0$, then it must be the case that $f_j'' < 0$ for the second-order condition to be verified. QED

ALTERNATIVE PROOF: Suppose goods 1 and 2 both have increasing marginal utility. In an interior solution, we have:

$$f_1'(x_1^*) = \frac{p_1}{p_2} f_2'(x_2^*)$$

Let $\Delta x_1 = +1$ and $\Delta x_2 = -p_1/p_2$. Total expenditure is unaffected and we now have:

$$f_1'(x_1^* + \Delta x_1) > \frac{p_1}{p_2} f_2'(x_2^* - p_1/p_2)$$

since $f_1'' > 0$ and $f_2'' > 0$. Hence, it is better to keep on reducing x_2 in favor of x_1 , i.e. $x_2 = 0$.

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1.55 ③

b) Normal good i : $\frac{\partial x_i}{\partial y} > 0 \Rightarrow \frac{\partial x_i}{\partial p_i} < 0$.Inferior good j : $\frac{\partial x_j}{\partial y} < 0$ and $\frac{\partial x_j}{\partial p_j} > 0 \Rightarrow \frac{\partial x_j}{\partial y}$ Let $f_1'' > 0$ and $f_j'' < 0$, $j = 2, \dots, n$.2 steps: ① Show that i and j cannot be both normal.
② Show that i must be normal.PROOF: Since $\alpha_1^* > 0$ and $\alpha_2^* > 0$, we initially have:

$$\frac{f_1'(\alpha_1^*)}{f_2'(\alpha_2^*)} = \frac{p_1}{p_2}.$$

① Assume that both are normal goods and that y increases by Δy , we have:

$$\Delta \alpha_1^* = \frac{\partial \alpha_1}{\partial y} \Delta y > 0 \text{ and } \Delta \alpha_2^* = \frac{\partial \alpha_2}{\partial y} \Delta y > 0.$$

$$\Rightarrow \alpha_1' \equiv \alpha_1^* + \Delta \alpha_1^* > \alpha_1^* \text{ and } \alpha_2' \equiv \alpha_2^* + \Delta \alpha_2^* > \alpha_2^* \text{ at } y' = y + \Delta y.$$

$$\Rightarrow f_1'(\alpha_1') > f_1'(\alpha_1^*) \text{ since } f_1'' > 0$$

$$f_2'(\alpha_2') < f_2'(\alpha_2^*) \text{ since } f_2'' < 0.$$

$$\Rightarrow \frac{f_1'(\alpha_1')}{f_2'(\alpha_2')} > \frac{p_1}{p_2} \quad \text{a contradiction.}$$

Hence, good 1 and any of goods $j = 2, \dots, n$ cannot be both normal.We can similarly show that good 1 and any of goods $j = 2, \dots, n$ cannot be both inferior.

1.55) b) ALTERNATIVE PROOF:

1.55 (3)

① We want to show that if $f_1'' > 0$ and $f_2'' < 0$, then both cannot be normal goods:

In an interior solution for x_1^* and x_2^* , we have:

$$\Psi \equiv p_2 f_1'(x_1^*) - p_1 f_2'(x_2^*) = 0.$$

Implicit differentiation yields:

$$\frac{\partial x_1^*}{\partial x_2^*} = - \frac{\Psi_{x_1}}{\Psi_{x_2}} = - \frac{p_2 f_1''(x_1^*)}{-p_1 f_2''(x_2^*)} < 0.$$

This says that if the price ratio remains constant, any increase in x_2^* must be accompanied by a decrease in x_1^* , or vice versa.

In particular, if good 1 is normal and thus increases with income, then good 2 must decrease with income.

QED

② It remains to be shown that good 1 is normal.

Assume not. Then, according to part a), goods $j=2, \dots, n$ are all normal. If y increases by Δy , we thus have:

$$\Delta x_1^* < 0 \quad \text{and} \quad \Delta x_j > 0, \quad \forall j=2, \dots, n.$$

(I don't figure out this proof. Intuitively I suspect it is wrong.)

c) $f_i'' < 0$, $i=1, \dots, n$.

Show that all goods are normal.

PROOF: First of all, all goods cannot be inferior, at the same time, lest the budget balancedness is violated.

It remains to be shown that one good cannot be inferior while another is normal.

Assume it is the case that good 1 is normal while good 2 is inferior. Following an increase in income Δy , we have

$$\Delta x_1^* = \frac{\partial x_1}{\partial y} \Delta y > 0 \text{ and } \Delta x_2^* = \frac{\partial x_2}{\partial y} \Delta y < 0.$$

$$\text{let } x_1' = x_1^* + \Delta x_1^* > x_1^* \text{ and } x_2' = x_2^* + \Delta x_2^* > x_2^*$$

$$\text{where } \frac{f_1'(x_1^*)}{f_2'(x_2^*)} = \frac{p_1}{p_2}.$$

$$\Rightarrow \frac{f_1'(x_1')}{f_2'(x_2')} < \frac{p_1}{p_2}. \quad \underline{\text{A contradiction.}}$$